

The Rayleigh–Taylor instability and internal waves in quantum plasmas

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Influence of quantum effects on the internal waves and the Rayleigh-Taylor instability in plasma is investigated. It is shown that quantum pressure always stabilizes the RT instability. The problem is solved both in the limit of short-wavelength perturbations and exactly for density profiles with layers of exponential stratification. In the case of stable stratification, quantum pressure modifies the dispersion relation of the inertial waves. Because of the quantum effects, the internal waves may propagate in the transverse direction, which was impossible in the classical case. A specific form of pure quantum internal waves is obtained, which do not require any external gravitational field.

Studies of quantum plasmas was initiated by Pines in the 1960's [1, 2], where the finite width of the electron wave function gives rise to dispersion, important in the high-density and/or low temperature regime. A number of quantum plasma studies has since appeared Ref. [3], *e.g.*, kinetic models of the quantum electrodynamical properties of nonthermal plasmas [4] and covariant Wigner function descriptions of relativistic quantum plasmas [5]. There has recently been a surge in the interest of quantum plasmas, see *e.g.* Refs. [6, 7, 8, 9, 10, 11, 12, 13], in particular the nonlinear properties of dense [14, 15, 16] or magnetized plasmas [13, 17, 18]. Many of these studies have motivated by new discoveries concerning nanostructured materials [19] and quantum wells [20], the discovery of ultracold plasmas [21, 22, 23], astrophysical applications [24], and inertial fusion plasmas [25]. For such quantum systems, the so called Bohm–de Broglie potential [6, 7, 8, 9, 10], as well as the zero temperature Fermi pressure [6, 7, 8, 9, 10] and other spin properties [11, 12, 13, 17, 18] can significantly modify the dynamics of the plasma. Moreover, quantum electrodynamical effects can give rise to completely new effects in plasma environments [26, 27, 28, 29] that can be of relevance in high-intensity quantum plasmas. Within the fluid approach to quantum plasmas [6, 7, 8, 9, 10, 11, 12, 27, 28, 29], collective effects can be described within a

unified picture.

The above examples mainly focuses on oscillations in homogeneous quantum plasma backgrounds. However, quantum plasmas can in practice often involve nonuniform density profiles, which often develop in an real (*e.g.* in astrophysics) or effective (*e.g.* in inertial confined fusion) external gravitational field. In the classical case, stratified plasma in a gravitational field inevitably exhibits either inertial waves or the Rayleigh-Taylor (RT) instability depending on whether the stratification is stable or unstable [30]. The purpose of the present paper is to study influence of the quantum effects on the internal waves and the RT instability. Here we show that quantum pressure always stabilizes the RT instability. The stabilization has a meaning of an effective "quantum velocity" reducing the instability growth rate. In that sense the stabilization is similar to the RT stabilization by an ablation flow in inertial confined fusion [31, 32, 33, 34]. We solve the problem in the limit of short-wavelength perturbations. We also find exact solutions to the RT stability problem for density profiles with layers of exponential stratification. In the case of stable stratification, quantum pressure modifies the dispersion relation of the inertial waves. Because of the quantum effects, the internal waves may propagate in the transverse direction, which was impossible in the classical case. Even more, we obtain specific form of pure quantum internal waves, which do not require any external gravitational field. The results could be of significance for astrophysical and inertial fusion plasmas.

We start with equations for continuity and momentum transport in quantum plasmas in the MHD approximation taking into account gravitational field [35, 36, 37, 38]

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \quad (1)$$

$$\begin{aligned} \rho \left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) u_l &= -\frac{\partial P}{\partial x_l} + \rho g_l + \\ &\frac{\hbar^2}{12m_e m_i} \frac{\partial}{\partial x_j} \left(\rho \frac{\partial^2}{\partial x_j \partial x_l} \ln \rho \right), \end{aligned} \quad (2)$$

where m_e , m_i are electron and ion masses. We consider small-scale effects taking the gravitational acceleration to be $\mathbf{g} = -g\hat{\mathbf{z}}$, where g is a constant. The hydrodynamic equilibrium is determined by the balance of forces

$$\frac{dP_0}{dz} = -\rho_0 g + \frac{\hbar^2}{12m_e m_i} \frac{d}{dz} \left(\rho_0 \frac{d^2}{dz^2} \ln \rho_0 \right) \quad (3)$$

However, the hydrodynamic equilibrium does not necessarily imply thermodynamic equilibrium. In the incompressible limit of an essentially subsonic plasma dynamics it allows both stable and unstable density profiles.

We consider small perturbations of the equilibrium according to $\varphi(x, z, t) = \varphi_0(z) + \tilde{\varphi}(z) \exp(i\omega t + ikx)$, where φ denotes any of the fluid variables, k is the perturbation wave number, and ω is the wave frequency. In the case of unstable stratification the frequency should be replaced by the instability growth rate $\sigma = i\omega$. Here we are interested in the incompressible plasma dynamics with

$$\frac{d\rho}{dt} = 0 \quad (4)$$

for any Lagrangian plasma parcel. Incompressible flow is typical both for the RT instability and the internal waves [39]; equation (4) holds for $\omega/k \ll c_s$, where $c_s = [(\partial P/\partial \rho)_S]^{1/2}$ is the sound speed. Then, to first order in the perturbed quantities, the system (1)–(2) reads

$$i\omega\tilde{\rho} + \tilde{u}_z \frac{d\rho_0}{dz} = 0, \quad (5)$$

$$\frac{d\tilde{u}_z}{dz} + ik\tilde{u}_x = 0, \quad (6)$$

$$i\omega\rho_0\tilde{u}_x = -ik\tilde{P} + ik\frac{\hbar^2}{12m_e m_i} \left\{ \frac{d}{dz} \left[\rho_0 \frac{d}{dz} \left(\frac{\tilde{\rho}}{\rho_0} \right) \right] - k^2 \tilde{\rho} \right\}, \quad (7)$$

and

$$i\omega\rho_0\tilde{u}_z = -\frac{d\tilde{P}}{dz} - \tilde{\rho}g + \frac{\hbar^2}{12m_e m_i} \frac{d}{dz} \left[\rho_0 \frac{d^2}{dz^2} \left(\frac{\tilde{\rho}}{\rho_0} \right) + \tilde{\rho} \frac{d^2}{dz^2} \ln \rho_0 \right] - k^2 \frac{\hbar^2}{12m_e m_i} \rho_0 \frac{d}{dz} \left(\frac{\tilde{\rho}}{\rho_0} \right). \quad (8)$$

We start with the limit of short wavelength perturbations, $k/\alpha \gg 1$, where $\alpha = d \ln \rho_0 / dz$ is the local inverse length scale of the density profile. In that case we can use the Wentzel-Kramers-Brillouin method [40] presenting perturbations in the form $\varphi(x, z, t) = \varphi_0(z) + \tilde{\varphi} \exp(ik_x x + ik_z z + i\omega t)$. Then, in the short wavelength limit Eqs. (6) – (8) reduce to

$$ik_z\tilde{u}_x + ik_x\tilde{u}_x = 0, \quad (9)$$

$$i\omega\rho_0\tilde{u}_x = -ik_x\tilde{P} - ik_xk^2\frac{\hbar^2}{12m_em_i}\tilde{\rho}, \quad (10)$$

$$\begin{aligned} i\omega\rho_0\tilde{u}_z &= -ik_z\tilde{P} - \tilde{\rho}g + \\ &k^2\frac{\hbar^2}{12m_em_i}\tilde{\rho}\left(\frac{1}{\rho_0}\frac{d\rho_0}{dz} - ik_z\right). \end{aligned} \quad (11)$$

Solving equations (5), (9) - (11) we find the dispersion relation for the internal waves in quantum plasma

$$\omega^2 = \omega_{cl}^2\frac{k_x^2}{k^2} + k_x^2\frac{\hbar^2}{12m_em_i}\left(\frac{1}{\rho_0}\frac{d\rho_0}{dz}\right)^2, \quad (12)$$

where the classical frequency of internal waves is designated by $\omega_{cl}^2 = -(g/\rho_0)d\rho_0/dz > 0$. In the case of "horizontal" internal waves, $k_z = 0$, $k_x = k$, the dispersion relation takes the form similar to electrostatic and electromagnetic plasma waves

$$\omega^2 = \omega_{cl}^2 + U_q^2k^2, \quad (13)$$

where ω_{cl} plays the same role as the plasma frequency and

$$U_q = \frac{\hbar}{2\sqrt{3m_em_i}}\left|\frac{1}{\rho_0}\frac{d\rho_0}{dz}\right| \quad (14)$$

is the characteristic quantum speed. Even in the case of zero gravity with $\omega_{cl} = 0$ we obtain specific quantum internal waves with the dispersion relation

$$\omega = U_qk_x. \quad (15)$$

In the case of unstable stratification $\mathbf{g} \cdot \nabla\rho < 0$, or

$$\frac{g}{\rho_0}\frac{d\rho_0}{dz} > 0, \quad (16)$$

we obtain the RT instability instead of internal waves. The RT perturbations of short wavelength are strongly localized within the layer with the most steep density profile $\max\alpha = d\ln\rho_0/dz$, e.g. see [31, 32], and the maximal instability growth rate corresponds to the mode with $k_z = 0$

$$\sigma = \sqrt{g\alpha - U_q^2k^2}. \quad (17)$$

As we can see from (17), quantum effects always play a stabilizing role for the RT instability. In a sense, this stabilization is similar to the effects of ablation flow and thermal conduction in the context of the inertial confined fusion [31, 32, 33, 34]. Still, the ablation flow may

be destabilizing for perturbations of long wavelength $k/\alpha \ll 1$ because of the additional Darrieus-Landau instability of a deflagration front [32, 34]. On the contrary, quantum effects are always stabilizing.

We can also find exact analytical solutions to Eqs. (5) - (8) for certain density profiles. For example, exact solution is possible for plasma consisting of one or several layers with exponential stratification $\rho_0 \propto \exp(\alpha z)$, $d(\ln \rho_0)/dz = \alpha = \text{const}$. After tedious but straightforward algebra, the system Eqs. (5) - (8) may be reduced to a single equation

$$\frac{d}{dz} \left(\rho_0 \frac{d\tilde{u}_z}{dz} \right) + \left(\frac{g}{\sigma_{\text{eff}}^2} \frac{d\rho_0}{dz} - \rho_0 \right) k^2 \tilde{u}_z = 0, \quad (18)$$

which has the same mathematical form as the respective classical result [39] but with the "effective" growth rate modified by the quantum terms

$$\sigma_{\text{eff}}^2 = \sigma^2 + U_q^2 k^2. \quad (19)$$

Therefore, reproducing calculations for the RT instability growth rate in the classical case with σ_{cl} , we find the respective result for quantum plasma as

$$\sigma^2 = \sigma_{cl}^2 - U_q^2 k^2. \quad (20)$$

Below we give some examples for the solution to Eq. (18).

1) Bounded plasma layer. We consider density profile $\rho_0 \propto \exp(\alpha z)$ within the layer $0 < z < L$ bounded by two rigid walls. The boundary conditions at the walls are $\tilde{u}_z = 0$. Then we have $\tilde{u}_z \propto \exp(\mu z)$ and Eq. (18) reduces to

$$\mu^2 \tilde{u}_z + \alpha \mu \tilde{u}_z \left(\frac{g\alpha}{\sigma_{\text{eff}}^2} - 1 \right) k^2 \tilde{u}_z = 0 \quad (21)$$

with the solutions

$$\mu_{1,2} = -\frac{\alpha}{2} \pm \left(\frac{\alpha^2}{4} + k^2 - \frac{g\alpha}{\sigma_{\text{eff}}^2} k^2 \right)^{1/2} \quad (22)$$

The boundary condition $\tilde{u}_z = 0$ at $z = 0, L$ is satisfied for

$$\left(\frac{g\alpha}{\sigma_{\text{eff}}^2} k^2 - \frac{\alpha^2}{4} - k^2 \right)^{1/2} = \frac{\pi n}{L}, \quad (23)$$

which leads to the instability growth rate

$$\sigma^2 = g\alpha \left(1 + \frac{\alpha^2}{4k^2} + \frac{\pi^2 n^2}{k^2 L^2} \right)^{-1} - U_q^2 k^2. \quad (24)$$

In the limit of short wavelength perturbations we recover our previous result Eq. (17); the mode number n has the physical meaning of the scaled wave number in the vertical direction, $k_z = \pi n/L$.

2) *A transitional layer.* We consider a transitional layer of width L separating two uniform plasmas of different density ρ_1 and $\Theta\rho_1 = \rho_1 \exp(\alpha L)$

$$\begin{aligned} \rho &= \rho_1 && \text{for } z < 0, \\ \rho &= \rho_1 \exp(\alpha z) && \text{for } 0 < z < L, \\ \rho &= \Theta\rho_1 && \text{for } z > L. \end{aligned} \quad (25)$$

In the uniform layers we obtain $\tilde{u}_z \propto \exp(\pm kz)$ from Eq. (18), which leads to the boundary conditions

$$\frac{d\tilde{u}_z}{dz} = \pm k\tilde{u}_z \quad (26)$$

at $z = 0, L$, respectively. Solving Eq. (18) with boundary conditions Eq. (26) we come to a transcendental equation

$$\frac{g\alpha}{2\sigma_{\text{eff}}^2} = 1 + \frac{\beta}{k} \coth(\beta L), \quad (27)$$

where

$$\beta = \left(\frac{\alpha^2}{4} + k^2 - \frac{g\alpha}{\sigma_{\text{eff}}^2} k^2 \right)^{1/2}. \quad (28)$$

Solution to Eq. (27) is unique and β is real for

$$k < k_c = \frac{\alpha}{2} \left(\sqrt{1 + 4/\ln^2 \Theta} - 2/\ln \Theta \right). \quad (29)$$

For $k > k_c$ the value β becomes imaginary, and we obtain multiple solutions to Eq. (27) similar to (24), see Fig. 1. Different branches of the plot at $k > k_c$ correspond to different mode numbers $n = 0 - 6$, which show the number of zeros for the eigenfunction \tilde{u}_z . In the limit of short wavelength perturbations, $k/\alpha \gg 1$, $kL \gg 1$, Eq. (27) goes over to (24). In the opposite limit of long wavelength perturbations, $k/\alpha \ll 1$, $kL \ll 1$, we find

$$\sigma^2 = \frac{\Theta - 1}{\Theta + 1} gk - U_q^2 k^2. \quad (30)$$

The first term in Eq. (30) is the RT instability growth rate at the discontinuity, while the second one stands for quantum stabilization. In principle, the term with quantum stabilization also contains small parameter $k/\alpha \ll 1$, and it should be omitted within the model of discontinuous density. However, the quantum term contains also another dimensionless parameter

$$\text{Fr}_q = \frac{U_q^2 \alpha}{g} = \frac{\hbar^2 \alpha^3}{12m_e m_i g}, \quad (31)$$

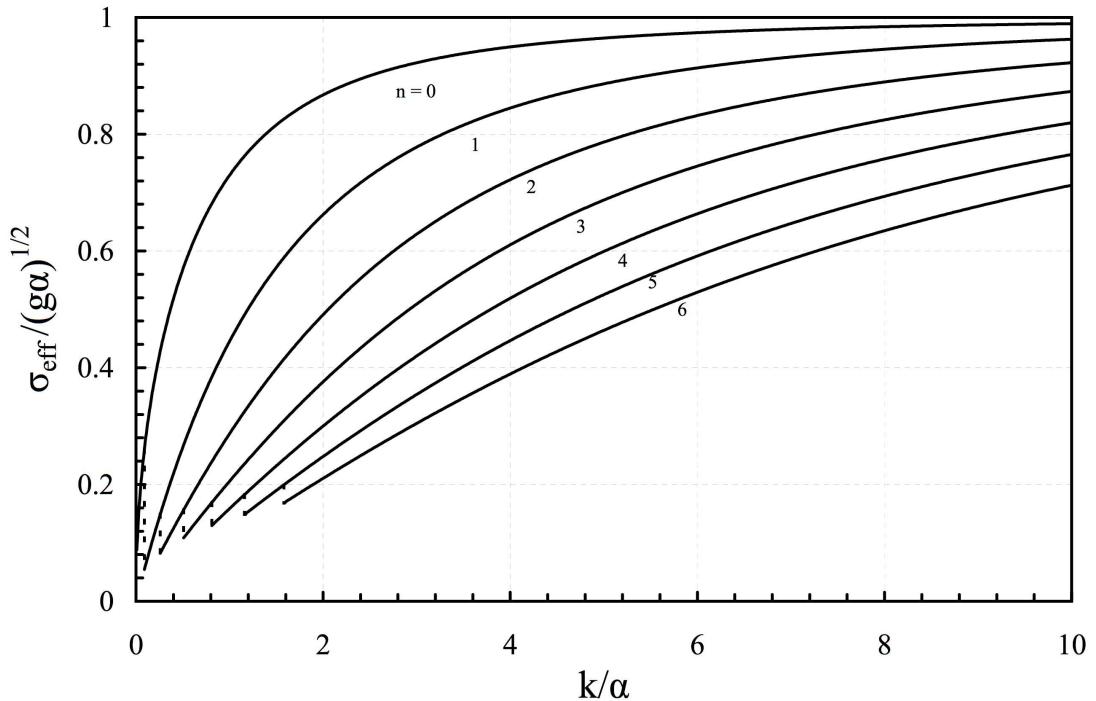


FIG. 1: Solution to Eq. (27), $\sigma_{\text{eff}}/\sqrt{g\alpha}$, versus the scaled wave number k/α for the density drop $\Theta = 8$. Different branches of the plot at $k > k_c$ correspond to different mode numbers $n = 0 - 6$.

which plays the role of a quantum Froude number, and which may be either large or small depending on a particular problem. For this reason, the quantum term in Eq. (30) may be important even in the limit of long wavelength perturbations. Numerical solution to Eq. (27) is shown in Fig. 2 for different values of the quantum Froude number. In the domain $k > k_c$ the solution is not unique; in Fig. 2 we presented the mode providing the maximal growth rate. As we can see from Fig. 2, in the case of the quantum Froude number above unity, $Fr_q > 1$, stabilization happens already for perturbations of short wavelengths, $k/\alpha < 1$.

The effect of quantum dispersion on the RT instability and internal waves in inhomogeneous systems can be of relevance in astrophysical environments [24] and ultracold plasmas [21]. The stabilizing effect due to the collective version of Heisenberg's uncertainty relation could even dominate the dynamics in very dense plasmas. However, for direct applications to such systems, more detailed calculations have to be made. Here we have indicated the principle dynamics of inhomogeneous fluids where such quantum effects can play a major role. Generalizations and more detailed calculations, incorporating *e.g.* magnetic pressure,

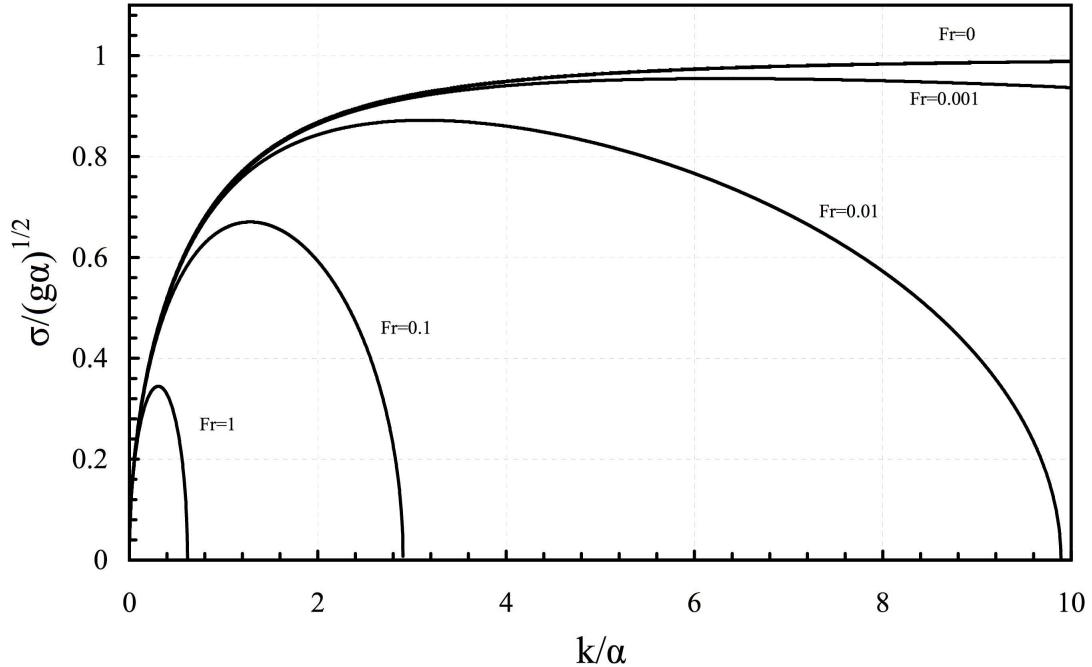


FIG. 2: Scaled instability growth rate $\sigma/\sqrt{g\alpha}$, Eq. (27), versus the scaled wave number k/α for the density drop $\Theta = 8$ and the quantum Froude number $\text{Fr}_q = 0 - 1$.

are left for future research.

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